

T H E
P R I N C I P A L P R O P O S I T I O N S

O F T H E

F I F T H B O O K O F E U C L I D, *K*

D E M O N S T R A T E D A L G E B R A I C A L L Y , A N D I L L U S T R A T E D B Y F I G U R E S .

F o r t h e U s e o f t h e S E N I O R S C H O L A R S o f R U G B Y S C H O O L .

B Y T . J A M E S , D . D .

H E A D M A S T E R o f R U G B Y S C H O O L , A N D L A T E F E L L O W o f K I N G ' s C O L L E G E , C A M B R I D G E .

C O V E N T R Y :

P R I N T E D B Y N . R O L L A S O N , I N H I G H - S T R E E T .

M , D C C , X C I .



P R E F A C E.

THE AUTHOR TO THE READER.

IF this little Production of mine should accidentally fall into the hands of any adept in Mathematics, it may be proper to inform him, that it was composed, not for publick sale, but for the private use of a School, not for the instruction of men, but of boys; and therefore if I should appear in any place too full of explanation, the learned reader may candidly attribute the offence to it's right cause, that is, to the desire which I have of being understood by all, and the necessity which I have experienced of speaking fully to young persons, in whom the quickness of apprehension must naturally vary.

I do not affect the character of a profound Mathematician, but I may perhaps know enough to enable me, upon conviction, to recommend to my Scholars so much at least of elementary Mathematics, as may be sufficient for comprehending the Laws of Nature by demonstration; or, in other words, for understanding scientifically the Four Branches of Natural Philosophy.

I am sensible that I am not daily conversant with Mathematical Studies, like the Tutors of our Universities, and therefore I have not a doubt, but that many parts of the following Production might be altered for the better. I profess, moreover, that I will be ready to avail myself of a well-meant hint from any quarter to amend what may be hastily written; for if I write at all for the benefit of my Pupils, the active occupation in life, in which I am engaged, will oblige me to write hastily.

I pro-

I profess that I can add nothing to the stock of Mathematical learning, but it may be possible to exhibit some demonstrations in a more convenient form than that in which they usually appear, and in a manner that may be more adapted to the youthful comprehension. - One thing only have I in view, which is, to allure young minds to a study, which must make them respectable, and may make them great. The want of some easy intelligible introduction to Mathematics has been, in effect, the ruin of many a young person in our Universities; because all studies are too often neglected, when the first object, the College-Lectures are not sufficiently understood for want of some preliminary knowledge of elementary Mathematicks. I present my young Student therefore with a Summary of the doctrine of PROPORTION: a doctrine of such consequence, that it is the main foundation of a right understanding of all Natural Philosophy; for what else is the scientific part of Natural Philosophy than the harmony of the Laws of Nature, or the PROPORTION observed between cause and effect?

Whoever has a right understanding of the Propositions here laid down, will find himself much assisted in comprehending the Fifth Book of Euclid as it stands either in Simpson or in Saunderson; and if any one should wish to avoid that trouble, he will find himself fully able to proceed to Euclid's Sixth Book, by the help of what is here delivered about Proportionals.

Rugby, July 23, 1791.

T. J A M E S.

I N T R O D U C T I O N.

Of P R O P O R T I O N.

SECT. 1. **WHAT PROPORTIONALS ARE.** When four numbers are of such a sort that the first has the same proportion to the second, that the third has to the fourth, then such Numbers are said to be proportional. Thus the four Numbers 2, 4; 3, 6, are proportionals, because AS 2 is the half of 4, SO 3 is the half of 6.

2. **THE CHARACTER EXPRESSING PROPORTIONALITY.** The mark and character whereby Proportionality is expressed is two dots interposed between the terms of the first pair, also two dots interposed between the terms of the second pair; but, between the two entire pairs, four dots are interposed: thus $2 : 4 :: 3 : 6$, and it is read thus, 2 is to 4 as 3 is to 6.

3. **OF A RATIO.** No comparison can arise from less than two Numbers; but two or a pair of Numbers is called a *Ratio* or Comparison, because in each pair we consider the ratio or relation which one of the two Numbers has to the other. Thus 2 is to 4 in a ratio of minority; and 4 is to 2 in a ratio of majority. Here we use the word ratio simply for *relation*: but ratio is also used to signify *the two Numbers related*, as taken together. Thus, in the proportion $2 : 4 :: 3 : 6$, the relation between the first two Numbers $[2 : 4]$ is called the *former Ratio*; and the relation between the last two $[3 : 6]$ is called the *latter Ratio*. In each Ratio, the first Number is called the *Antecedent*, and the second Number is called the *Consequent*. Thus, in the proportion $2 : 4 :: 3 : 6$, the numbers 2 and 3 are Antecedents (from *ante* and *cedo*) because they *go first* in each Ratio; and 4 and 6 are Consequents (from *consequor*) because they *follow after* their Antecedents.

4. **TWO EQUAL RATIOS MAKE A PROPORTION.** When we consider four Numbers together, as 2, 4; 3, 6, and compare them by pairs, first comparing together the former two, and next the latter two; and find that the ratio or comparison between one Pair is *equal* to the ratio between the other Pair, we call this equality of Ratios a PROPORTION. Thus, comparing the former Pair or Ratio $[2 : 4]$ with the latter Ratio $[3 : 6]$, I find,

that the former Ratio = latter Ratio
or, that the Ratio of $2 : 4$ = ratio of $3 : 6$

because

because the Comparison resulting from either Ratio is the same, since each Antecedent is the half of it's Consequent. Such an equality of two Ratios makes a Proportion, *quasi a pro portione*, because in respect of portion or part each Antecedent is the same part of it's Consequent.

5. DEFINITION OF A RATIO. Ratio is the *relation* which quantities of the *same kind* have to one another. Thus 2 Units is the half of 4 Units; but we cannot say that 2 hours is the half of 4 miles, because no idea of the relation between 2 hours and 4 miles can exist, since hours and miles are not quantities of the same kind; and therefore 2 hours and 4 miles cannot be compared together, in respect to the excess or defect of the one above the other; for as an hour is no part of a mile, so 2 hours is no half of 4 miles; though the numbers 2 and 4, affixed to the hours and miles, may be compared, if taken alone or abstractedly, or without any application and reference to hours and miles; but 2 and 4, so considered, are units or things of the same kind. So also in Saunderson's question 19, page 10, we seek in the operation for a proportion that shall exist between four sets of units, and it is found to exist in the following four sets, viz.

$$\begin{array}{cccc} \text{dwts.} & D. & \text{dwts.} & D. \\ 20 & : 66 & :: & 805 : 2656.5 \end{array}$$

Here is a true proportion in point of Units, so that 20 (considered as a number, or as so many Units) is the same part of 66, that 805 is of 2656.5 (viz. a 3. 3 part;) but though a ratio, or relation exists between 20 and 66 as Units, so that in this sense 20 be a part of 66, yet no ratio, no relation at all exists between them as being things of different kinds, for 20 d.wts are no part whatever of 66 pence, because a d.wt is no part of a penny. Hence in Quest. 19, and in all others of the Rule of Three Direct, we compared the 1st and 3d terms together, and so also the 2d and 4th, to express the Proportion briefly in words, by saying, *the Pennyweights are as the Pence*; whence the four terms of the Proportion were thus changed;

$$\begin{array}{cccc} \text{dwts.} & \text{dwts.} & D. & D. \\ 20 & : 805 & :: & 66 : 2656.5 \end{array}$$

Here an idea results from comparing the two sets of d.wts together, namely, that 20 d.wts is a certain part of 805 d.wts (it will be found a 40-th $\frac{1}{4}$ part of 805) and the like idea results from comparing the two sets of pence together, that 66 d. is a certain part of 2656.5 d., namely a 40-th $\frac{1}{4}$ part.

Now that the four first Terms as stated above in the order in which they stood for the operation [viz. $\begin{array}{cccc} \text{dwts.} & D. & \text{dwts.} & D. \\ 20 & : 66 & :: & 805 : 2656.5 \end{array}$] may be so changed as to give the order in which they have last stood [viz. $\begin{array}{cccc} \text{dwts.} & \text{dwts.} & D. & D. \\ 20 & : 805 & :: & 66 : 2656.5 \end{array}$] for the sake of comparing two things

things of like kinds together (namely d.wts with d.wts and pence with pence) will be demonstrated in the 6th Prop. of our 5th Book of Euclid.—Since then things of the *same* kind only can be compared together, in respect to excess or defect, and since these alone have any ratio or relation to each other, so that the one can be said to be less than the other, or greater, or equal to the other, it will be necessary to define what Quantities or Magnitudes are of the same kind, or what are Homogeneous Quantities, which are of an ὁμοῖος *similis, par, idem, γένος* genus.

6. WHAT ARE HOMOGENEOUS QUANTITIES. When the less of two Quantities compared is capable of being so multiplied as to exceed the greater, then those two are homogeneous Quantities. Thus a yard in length can be so multiplied as to exceed a mile in length, but it cannot be so multiplied as to exceed a year in duration; because a yard and a year are heterogeneous Quantities; therefore a yard hath a certain ratio to a mile, but it hath none to a year. For a like reason, lines can have no ratios to surfaces, nor surfaces to solids; but lines are to be compared with lines, surfaces with surfaces, and solids with solids. This is Def. 4. of the 5th Book of Euclid.

7. WHAT IS MEANT BY THE WORD PART, AND PARTS, AND A MULTIPLE. By *part* is meant an aliquot part, or such a part as being taken a certain number of even or integral times will make up the whole. Thus 3 is a part or an aliquot part of 12, in Euclid's sense, because 3 being taken 4 times will exactly make up 12; or in other words, 3 divides and measures 12 exactly; and though 9 be a part of 12 in that sense in which a part is distinguished from the whole, yet 9 in Euclid's sense is not a part but parts of 12, as being three fourth parts of it. It may now be perceived that 9 is an aliquant part of 12, because it is *aliquantum* somewhat of 12, and can never, by being taken a certain number of times make up 12. It is true that 9 multiplied by $1\frac{1}{3}$ makes up 12, but in this case the multiplicator expresses not *even*, or integral times, but a time and a fraction of a time.—Aliquot is derived from *aliquoties* several times, certain times.—Hitherto we have explained the word PARTS as if they were always less than the whole, but they may be also greater than the whole. Thus 13 may be called parts of 12, namely, such parts as contain 12 one time and a fraction of a time, or $1\frac{1}{12}$. So 33 may be called parts of 12, namely, such containing parts as comprehend 12 twice and $\frac{9}{12}$ or $\frac{3}{4}$ of it besides. Further, 24 may be also called parts of 12, namely such parts of 12 as make it up twice; but whenever the parts exactly make up the number spoken of any number of exact or integral times (excluding here any fraction of a time) we call such parts a *Multiple*, or that which contains exactly integral multiplications of a number. Thus 24 and 36 may be either called *parts* of 12, or *multiples* of 12; for 24 contains two multiplications of 12, and 36 contains three such.—To sum up what has been said, a *part*, in Euclid's sense, is that aliquot part which measures the whole any number of times evenly, and without a fraction. That which is not thus a part, must be *Parts* of the whole; for Parts are either less
or

or greater than the whole: if less than the whole, they must be parts of some denomination or other, and may be expressed by some proper Fraction as $\frac{3}{4}$, $\frac{2}{5}$, $\frac{5}{8}$, $\frac{9}{10}$, &c. *ad infinitum*; but, if the Parts be greater than the whole, they are still *parts* in the sense in which improper fractions (which contain more than a whole) are said to consist of such *parts* as their Denominator expresses. In this sense we have said above, that 13 was parts of 12, namely, such parts of 12 as the improper Fraction $\frac{13}{12}$ expresses, or $1\frac{1}{12}$; and so also we called 33 such parts of 12, as the improper fraction $\frac{33}{12}$ expresses, or $2\frac{9}{12} = \frac{3}{4}$; and, lastly, in this sense also, in the proportion produced in Sect. 5, the Antecedents are parts of their Consequents. If the Parts contain any exact number of wholes, then they may be called not only *Parts* but *Multiples*, as $\frac{24}{12} = 2$, and $\frac{36}{12} = 3$.

Since then if two things be compared together (as they are in a Ratio) there can be only, from the nature of things, three considerations in the comparison, viz. that the one of the two is less than the other, or greater, or equal to the other; or, in other words, that the one is an exact part, or parts, of the other, or equal to the other, let us seek to find in all Ratios, what part or parts an Antecedent is of it's Consequent.

8. HOW TO FIND WHAT PART OR PARTS AN ANTECEDENT IS OF IT'S CONSEQUENT. Whether an Antecedent be less or greater than it's Consequent, its comparative magnitude will be found by writing those two fraction-wise, viz. by writing the Antecedent as a Numerator, and the Consequent as a Denominator. Thus in the ratio of 3 : 4 I say that 3 is $\frac{3}{4}$ of 4 or that the Numerator is such parts of it's Denominator as the total fraction expresses. For having written the ratio of 3 : 4 fraction-wise as $\frac{3}{4}$, I take the two members of the Fraction separately, and consider 3 as 3 Integers or Units (which it is, if taken separately) and 4 as 4 Units: then interposing the total fraction itself between those separate Units, I affirm, that 3 Units is $\frac{3}{4}$ of 4 Units; for certainly 1 Unit is $\frac{1}{4}$ of 4 Units; and 2 Units is $\frac{2}{4}$ of 4 Units; and therefore 3 Units must be $\frac{3}{4}$ of 4 Units.—In like manner every Numerator will be such part or parts of it's Denominator as the total fraction expresses. Thus in $\frac{12}{8}$ the Nr. 12 or 12 Units is $\frac{12}{8}$ of it's Dr. 8 Units; for certainly 1 Unit is $\frac{1}{8}$ of 8 Units; 2 Units is $\frac{2}{8}$ of 8 Units; and therefore 12 Units must be $\frac{12}{8}$ of 8 Units; or, briefly, 12 is $\frac{12}{8}$ of 8. But $\frac{12}{8} = \frac{3}{2}$; therefore also 12 is $\frac{3}{2}$ of 8. Lastly, the fraction $\frac{12}{4}$ shews, that the Nr. 12 Units is $\frac{12}{4}$ of it's Dr. 4 Units; for certainly 1 Unit is $\frac{1}{4}$ of 4 Units; 2 Units is $\frac{2}{4}$ of 4 Units; and therefore 12 Units must be $\frac{12}{4}$ of 4 Units; or, briefly, 12 is $\frac{12}{4}$ of 4. But $\frac{12}{4} = 3$; wherefore the fraction $\frac{12}{4}$ shews also, that 12 is exactly 3 times 4, or is a Multiple of 4.

Thus we have proved in the Ratio of 12 : 8, by writing it fraction-wise, that 12 is $\frac{12}{8}$ of 8, and hereby we have expressed what parts the Antecedent is of it's Consequent. So also in the Ratio of 12 : 4, by writing it fraction-wise, we have proved, that the Antecedent 12 is $\frac{12}{4}$ of it's Consequent 4.

9. Since

9. Since then we can thus find out, what part or parts every Antecedent or Nr. is of it's Consequent or Dr.; and since, if, in two Ratios, both Antecedents be equally great with respect to their Consequents, we have then a true Proportion, it follows, that

TWO EQUAL FRACTIONS MAKE A PROPORTION,

because whatever part or parts the one Nr. is of it's Dr., if the fractions be equal, the other Nr. is the same part or parts * of it's Dr.; for otherwise the two fractions could not express the same thing. Thus in the two fractions $\frac{5}{20}$ and $\frac{25}{100}$, the former $\frac{5}{20} = \frac{1}{4}$, and the latter also $\frac{25}{100} = \frac{1}{4}$, whence the two fractions are equal, so that 5 is $\frac{5}{20}$ (or $\frac{1}{4}$) of 20, and 25 is $\frac{25}{100}$ (or $\frac{1}{4}$) of 100; consequently we have two equal Ratios or a Proportion, arising from the two equal Fractions $\frac{5}{20}$ and $\frac{25}{100}$, viz.

$$5 : 20 :: 25 : 100 \text{ OR, Nr. : Dr. :: Nr. : Dr.}$$

So also, from the equal fractions $\frac{1}{4}$ and $\frac{5}{20}$, we have, $1 : 4 :: 5 : 20$

And, from the equal fractions $\frac{1}{4}$ and $\frac{25}{100}$, we have, $1 : 4 :: 25 : 100$

Thus any two equal fractions make two equal Ratios or a true Proportion; and on the other side, if a Proportion be true, it follows, that it's two equal Ratios make two equal fractions. Here then is a criterion and test of Proportionality in Numbers, namely, by examining the equality of two Ratios, and seeing whether they be two equal fractions.—This is one test of Proportionality: We shall hereafter find another.

The Postulatum and Definitions prefixed to our 5th Book of Euclid will now be easily understood.—But before we proceed, it will be right to advertise the reader, that both Saunderson's and Simson's Propositions of Euclid are referred to in the margin of our leaf. Thus in our Prop. 2, page 11, it is signified in the margin by *el. 6. Si. 16*, that the Proposition of the elements of Euclid, which corresponds to our Prop. 2, is the 16th Prop. of the 6th Book of Euclid, as published by Simson. So also in our Prop. 4, page 13, it is signified in the margin by *el. 5. Sa. Si. 7*, that the corresponding Proposition to our Prop. 4 is the 7th Prop. of the 5th Book of the elements of Euclid as published both by Saunderson and Simson.—Def. is used for Definition; and Cor. for Corollary.

* From this expression a Proportion may be considered as thus derived, *quasi a proportionibus*, because, in respect of portions or parts, the two Antecedents are the same part or parts of their Consequents. See Section 4.

E U C L I D. B O O K V.

P O S T U L A T U M.

HOMOGENEOUS QUANTITIES ARE EQUAL; or the one is *greater* or *less* than the other.—See the *Introduction*, Sect. 6. and end of Sect. 8.

D E F I N I T I O N S.

- (*el. 5. Sa. Si. Def. 1.*) I. A *less* quantity is a part of a greater, when the less (taken a certain number of times) will exactly make up the greater. Thus a foot is a part of a yard.—See the *Introduction*, Sect. 7.
- (*el. 5. Sa. Si. Def. 2.*) II. A *greater* quantity is a multiple of a less, when it contains the less a certain number of times exactly. Thus a yard is a multiple of a foot.—*Introduct. Sect. 7.*
- (*el. 5. Sa. Si. Def. 3.*) III. Ratio is the *relation* between two homogeneous quantities compared together. Thus $a : b :: 2 : 4$. Either the two first terms or the two last express the relation between the other two. Therefore the two first terms are called the *former* Ratio; and the two last the *latter* Ratio. In each Ratio the first Term is called *Antecedent*, the latter *Consequent*.—*Introduct. Sect. 5. and 6.*
- (*el. 5. Sa. Si. Def. 6. and 8.*) IV. Two equal Ratios form a Proportion.—*Introduct. Sect. 4.*
- Note.* From the sameness of its two Ratios, a Proportion is also called an *Analogy*, because the same Relation is repeated, as the derivation implies, from *ανα* again, and *λογος* ratio.
- (*el. 5. Sa. Si. Def. 9.*) V. Three different terms may form a Proportion. Thus $2 : 4 :: 4 : 8$. Here the second term is called a *mean Proportional* between the first and third. And as the succeeding terms (2, 4, 8) increase in the same ratio, by the same Multiplier (2,) and, being read backwards, decrease in the same ratio by the same Divisor (2,) they are called *continued* proportionals; because the ratio between any two continues the same throughout.

P R O P O.

P R O P O S I T I O N I.—See *Introduct. Sect. 9.*

(el. 7. Def. 20.)

IF two Fractions be equal, their terms will form a proportion: so that Nr. : Dr. :: Nr. : Dr. Thus if $\frac{2}{4} = \frac{3}{6}$ (because either of them $= \frac{1}{2}$) then $2 : 4 :: 3 : 6$, or the first is to the second, as the third is to the fourth; so that the two ratios (or comparisons) are equal, and therefore will form a proportion (by Def. 4.)

For two *equal* fractions express the same thing. Therefore if the one Nr. be half it's Denominator; then the other Nr. is also half of it's Dr.—and so if the Nrs. be any other part, when the fractions are *equal*, such Nrs. are constantly the same part of their Drs.

If the first Nr. = it's Dr.; then also the second Nr. = it's Dr. Thus $\frac{4}{4} = \frac{8}{8}$ (for both = 1) in which case $4 : 4 :: 8 : 8$. If the first Nr. be greater than it's Dr. then also the second Nr. is greater than it's Dr. Thus $\frac{10}{5} = \frac{12}{6}$ (for both = 2) and $10 : 5 :: 12 : 6$. Thus (if the fractions be equal) the first Antecedent cannot be *less* than, *equal* to, or *greater* than it's Consequent, but at the same time the second Antecedent will in like manner be *less* than, *equal* to, or *greater* than it's Consequent.—Therefore, in each of the cases above-stated, the two Ratios (or comparisons) correspond altogether, and are exactly alike; consequently (by Def. 4) they form a proportion. Hence, if $\frac{a}{b} = \frac{c}{d}$ then $a : b :: c : d$, or the first is to the second, as the third to the fourth.

Cor. 1. If $2 : 4 :: 3 : 6$
then $\frac{2}{4} = \frac{3}{6}$

Cor. 1. A proportion forms two equal fractions; as if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$

P R O P O S I T I O N II.

(el. 6. Si. 16.)

If $2 : 4 :: 3 : 6$ Then $2 \times 6 = 4 \times 3$

If four quantities be proportionable; the product of the extremes will equal the product of the means: that is if $a : b :: c : d$, then shall $ad = bc$

For since $a : b :: c : d$ (by sup.); therefore (by Pr. 1. Cor. 1.) $\frac{a}{b} = \frac{c}{d}$; whence multiplying the first fraction by d , and the second by b , we have $\frac{ad}{bd} = \frac{bc}{bd}$; and by multiplying these last fractions by bd , we have $ad = bc$.

Cor. 1.

Cor. 1.—*Rule Direct.*
 If $2 \times 6 = 4 \times 3$
 Then $6 = \frac{4 \times 3}{2}$

Rule Inverse.
 If $2 : 4 :: 3 : 6$ and the
 three first Numbers be
 stated, 3, 4, 2, then,
 as before, $6 = \frac{3 \times 4}{2}$

Cor. 2. (*el. 5. Sa. Si. 9.*)
 First, if $12 : 6 :: 3 \times 4 : 6$
 then $12 = 3 \times 4$
 Secondly,

$$\frac{c}{\quad} = \frac{a}{b}$$

(*el. 6. Si. 16.*)
 If $2 \times 6 = 4 \times 3$ then making
 2 and 6 extremes, we have
 $2 : 4 :: 3 : 6$. Also $2 : 3 :: 4 : 6$.
 Also $6 : 4 :: 3 : 2$. Also $6 : 3 :: 4 : 2$.

Cor. 1. Hence you may demonstrate the Rule of Three, which consists in finding a fourth proportional; for since $ad = bc$, divide both sides of the Equation by a , and you have $d = \frac{bc}{a}$, which is as much as to say, that if three Numbers, a, b, c , be given, a fourth proportional d may be obtained by multiplying the second and third Numbers together, and dividing the product by the first.

In the Rule of Three Inverse, which also consists in finding a fourth proportional, the right order of the three first of the four proportionals, a, b, c, d , is inverted for the purposes of the operation, so that these three terms when stated for the operation, become c, b, a ; whereby a , though really the *first* term in the true proportion, becomes the *third* term as stated for the operation. Thus a , the first in the true Proportion, is still as before, the dividing Number, and d is still found as before, by dividing bc by a , in respect of the true proportion; but in respect of the order in which a stands for the operation [c, b, a] a is the third Term, and b, c , are the second and first; therefore, in reference to this statement, d is found by dividing the second and first Terms by the Third.—Thus the terms of the true Proportion are inverted only in their statement for the operation; and therefore such statement must be inverted or read backwards to form a true proportion.

Cor. 2. If two quantities a and b have both the same ratio to any third as c ; or if c hath the same ratio to both a and b ; in either of these cases a and b must be equal to each other.

First, since $a : c :: b : c$ (by sup.); therefore $ac = bc$ (by Prop. 2); and dividing both sides by c , we have $a = b$.

Secondly, if $c : a :: c : b$ then $a = b$. For (by Prop. 2) $cb = ac$, whence $b = a$.

PROPOSITION. III.—[*This Prop. is the Converse* of Prop. 2.*]

If the product of two quantities on one side of an equation be equal to the product of two quantities on the other; such an equation may be resolved into a proportion, viz. by making the two quantities on either side the extremes, and those on the other side the middle terms: that is if $ad = bc$, then $a : b :: c : d$.

* The converse of a Proposition *proves* that which was before *supposed*. Thus the Case, *quasi convertitur*, is as it were turned about and changed; for a former *supposition* is turned into a *Proof*.—Prop. 2 and Prop. 3 are the Converse of each other.

For

Again by making 4 and 3 extremes,
we have $4 : 2 :: 6 : 3$. Also
 $4 : 6 :: 2 : 3$. Also beginning
with 3, $3 : 2 :: 6 : 4$. And
 $3 : 6 :: 2 : 4$.

Cor. 1.—(el. 6. Si. 17.)
If $2 : 4 :: 4 : 8$
then $2 \times 8 = 4 \times 4$

For, if d be not in proportion, let e . Then since (by sup.) $a : b :: c : e$, this e must be either greater or less than d . But it may be proved equal to it. For $a : b :: c : e$ (by sup.); therefore, $ae = bc$ (by Prop. 2.) $= ad$ (by sup.) therefore $ae = ad$; and dividing both sides by a , $e = d$. Therefore whatever be the supposed quantity (as e) in just proportion, such quantity must equal the given quantity d .

Cor. 1. If the two middle terms are alike, (see our Def. 5) that is, if $a : b :: b : c$, then is $a \times c = b \times b$; that is, the product of the extremes is equal to the square of the means; and conversely, if $ac = bb$ then is $a : b :: b : c$, [and consequently by Cor. 1. of Prop. 1. $\frac{a}{b} = \frac{b}{c}$]

Note. Prop. 1. exhibits one criterion of proportionality, viz. if four Quantities make two equal Fractions, then have we four Quantities in a true proportion.—Prop. 2 and 3 exhibit another and a new criterion of proportionality, viz. if in four Quantities the multiplication of extremes and means is equal, then do those four Quantities make a true Porportion.—Hereafter we may take either equation, viz. $\frac{a}{b} = \frac{c}{d}$ or $a \times d = b \times c$ for the mark of proportionality; one equation implying the other.—By the help of these two tests, which will constantly be referred to, we shall demonstrate all the remaining Propositions, concerning the Properties of Proportionality.

P R O P O S I T I O N IV.

If two equal quantities a and b be compared with a third as c , I say then, that both a and b will have the same ratio to c ; and *vice versa*, that c will have the same ratio both to a and b .

First, if $a = b$, take any quantity c ; then $a : c :: b : c$. For since $a = b$ (by sup.), divide both sides by c , whence $\frac{a}{c} = \frac{b}{c}$; therefore (by Pr. 1) $a : c :: b : c$.

Secondly, if $4 = 4$, then $\frac{1}{4} = \frac{1}{4}$
and $1 : 4 :: 1 : 4$

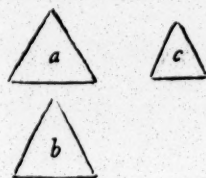
Secondly, if $a = b$, then c will have the same ratio to them both, so that $c : a :: c : b$. For since $a = b$, make them both Drs. with the same Nr. c . Then since $\frac{c}{a} = \frac{c}{b}$ because both Nrs. and Drs. are equal, therefore $c : a :: c : b$ (by Pr. 1)

D.

P R O P O-

(el. 5. Sa. Si. 7.)

First,



Secondly, if $4 = 4$, then $\frac{1}{4} = \frac{1}{4}$
and $1 : 4 :: 1 : 4$

(el. 5. Sa. 4. Cor.)

(el. 5. Si. Prop. B. p. 121)

If $2 : 4 :: 3 : 6$ then $4 : 2 :: 6 : 3$ *invertendo*

(el. 5. Sa. Si. 16.)

If $10 : 5 :: 12 : 6$ then $10 : 12 :: 5 : 6$ *alternando or permutando*

P R O P O S I T I O N V.

If four quantities be proportionable; they will also be inversely proportionable: as if $a : b :: c : d$, then $b : a :: d : c$.—See el. 5. Sa. Si. Def. 14.

For since $a : b :: c : d$ (by sup.); therefore $b \times c = a \times d$ (by Pr. 2); whence (by Pr. 3) making b and c extremes, we have $b : a :: d : c$.

P R O P O S I T I O N VI.

If four *homogeneous* quantities be proportionable, they will also be alternately proportionable; as if $a : b :: c : d$, then $a : c :: b : d$.—See el. 5. Sa. Si. Def. 13.

For since $a : b :: c : d$ (by sup.); therefore (by Pr. 2) $a d = c b$; whence (Pr. 3.) $a : c :: b : d$.

To estimate the proportionality of the four Quantities taken alternately by our criterion of proportion, namely, by dividing the 1st. by the 2d., or by finding what part or parts the 1st. is of the 2d.,—to do this, observe that the 1st. and 3d. and 2d. and 4th. must be of the same kind; for otherwise, after alternation, the first cannot be said to be any *part* (or parts) of the second, and so our criterion would not be applicable. Abstract Numbers or collections of Units are all of the *same kind*, and therefore no restriction is required for them as stated in the margin of our page, for here no name is given to the terms of the Proportion beyond that of Units: but if we look into the Questions of the Rule of Three, and if we would, as we do there, compare two things of a certain name or kind, with two other things of a different kind, and so give expressions to the two comparisons or the two Ratios of the proportion, then the terms of the former Ratio must be of the *same kind*, and such also must be the terms of the latter Ratio.—See Introduct. Sect. 5.

P R O P O S I T I O N VII.

(el. 5. Sa. 17.)

If $10 : 5 :: 12 : 6$ then *dividendo* $10 - 5 : 5 :: 12 - 6 : 6$

If four quantities be proportionable, the excess of the *first above the second* is to the second; as the excess of the *third above the fourth* is to the fourth: thus if $a : b :: c : d$, then $a - b : b :: c - d : d$.

See el. 5. Sa. Si. Def. 16.

For by the nature of Algebraic multiplication $\overline{a - b} \times d = ad - bd$ } But these latter sides are
 Also, for the same reason, $b \times \overline{c - d} = bc - bd$ } equal

equal to one another; for since (by sup.) $a : b :: c : d$, therefore (by Pr. 2) $ad = bc$. Hence the two former sides are equal, viz. $\overline{a-b} \times d = b \times \overline{c-d}$, whence (Pr. 3) $a-b : b :: c-d : d$.

Cor. 1.

If $2 : 6 :: 3 : 9$
then *dividendo*
 $2 : 6 - 2 :: 3 : 9 - 3$

So also the first is to the excess of the *second above the first*; as the third is to the excess of the *fourth above the third*; as if $a : b :: c : d$, then $a :: b - a : c : d - c$.

For $a \times \overline{d-c} = ad - ac$
Also $b - a \times c = bc - ac$ } Therefore $a \times \overline{d-c} = \overline{b-a} \times c$, whence (Pr. 3) $a : b - a :: c : d - c$.

PROPOSITION VIII.

If four quantities be proportionable, the first is to the *sum of the first and second*; as the third is to the *sum of the third and fourth*: thus if $a : b :: c : d$, then $a : a + b :: c : c + d$.

If $2 : 4 :: 3 : 6$
then *componendo*
 $2 : 2 + 4 :: 3 : 3 + 6$

For $a \times \overline{c+d} = ac + ad$
Also $\overline{a+b} \times c = ac + bc$ } Therefore $a \times \overline{c+d} = \overline{a+b} \times c$, whence (Pr. 3) $a : a + b :: c : c + d$

Further, *invertendo* (Pr. 5) we have $a + b : a :: c + d : c$.

Again, (el. 5. Sa. Si. 18.)

If $2 : 4 :: 3 : 6$
then *componendo*
 $2 + 4 : 4 :: 3 + 6 : 6$

Again, if $a : b :: c : d$, then $a + b : b :: c + d : d$.—See el. 5. Sa. Si. Def. 15.

For $\overline{a+b} \times d = ad + bd$
Also $b \times \overline{c+d} = bc + bd$ } Therefore $\overline{a+b} \times d = b \times \overline{c+d}$, whence (Pr. 3) $a + b : b :: c + d : d$

Further, *invertendo* (Pr. 5.) we have $b : a + b :: d : c + d$.

(el. 5. Sa. 19. Scholium)

(el. 5. Si. Prop. E. page 136.)

If $10 : 5 :: 12 : 6$
then *convertendo*

$10 : 10 - 5 :: 12 : 12 - 6$

PROPOSITION IX.

If four quantities be proportionable, the first is to the *excess of the first above the second*; as the third is to the *excess of the third above the fourth*. Thus if $a : b :: c : d$, then $a : a - b :: c : c - d$. See el. 5. Sa. Si. Def. 17.

For $a \times \overline{c-d} = ac - ad$
Also $\overline{a-b} \times c = ac - bc$ } Therefore $a \times \overline{c-d} = \overline{a-b} \times c$, whence (Pr. 3.) $a : a - b :: c : c - d$.

PROPO.

(Saunderson's 12thly, page 95.)

PROPOSITION X.—[This Proposition is not in Euclid, but it is useful.]

If $10 : 5 :: 12 : 6$
then *miscendo terminos*
 $10 + 5 : 10 - 5 :: 12 + 6 : 12 - 6$

If four quantities be proportionable, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

Thus if $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

For $\overline{a + b} \times \overline{c - d} = ac - ad + bc - bd = (\text{Pr. 2.}) ac - bd$
Also $\overline{a - b} \times \overline{c + d} = ac + ad - bc - bd = (\text{Pr. 2.}) ac - bd$ } Therefore
 $\overline{a + b} \times \overline{c - d} = \overline{a - b} \times \overline{c + d}$, whence $a + b : a - b :: c + d : c - d$ (by Pr. 3.)

(el. 5. Sa. Si. 11.)

PROPOSITION XI.

$2 : 4 :: (3 : 6 ::) 10 : 20$

If two ratios be equal to a third, they must be equal to one another: as if $a : b :: c : d$ and $c : d :: e : f$ then $a : b :: e : f$ (Def. 4.)

ex æqualitate rationis interpositæ.

For, since $a : b :: c : d$ (by sup.) therefore $\frac{a}{b} = \frac{c}{d}$ (by Cor. 1. Pr. 1.) } whence $\frac{a}{b} = \frac{e}{f}$ and
Also, since $c : d :: e : f$ (by sup.) therefore $\frac{e}{f} = \frac{c}{d}$ (by Cor. 1. Pr. 1.) } (Pr. 1.) $a : b :: e : f$.

(el. 5. Sa. Si. 12.)

PROPOSITION XII.

Addition of equal ratios.

If $2 : 4$
and $3 : 6$:: $10 : 20$

Then $2 + 3 : 4 + 6 :: 10 : 20$

If there be any number of equal ratios, as one of the antecedents is to it's consequent, so is the sum of all the antecedents to the sum of all the consequents.

Thus, if $a : b :: c : d :: e : f$, then $a + c + e : b + d + f :: a : b$.

Equal ratios may be also *subtracted*; as appears from the same process of demonstration by using the sign — instead of +.

For $\overline{a + c + e} \times b = ab + bc + be$ } But these latter sides are equal, because since (by sup. $a : b :: c : d$,
Also $\overline{b + d + f} \times a = ab + ad + af$ } therefore (Pr. 2.) $ad = bc$. Also $af = be$, for
since (by sup.) $a : b :: (c : d ::) e : f$, therefore
(by Prop. 11.) $a : b :: e : f$, whence (Pr. 2.) $af = be$. Hence $\overline{a + c + e} \times b = \overline{b + d + f} \times a$;
wherefore (Pr. 3.) $a + c + e : b + d + f :: a : b$.

PROPO-

(*el. 5. Sa. Si. 22.*)

Ex æqualitate rationum ordinate,
or *ex æquo ordinate,*
or *ex æquali.*

If 10, 20, 40 be one series
and 12, 24, 48 be another series
And if $10 : 20 :: 12 : 24$
also $20 : 40 :: 24 : 48$

Then, drawing a line through 20
and 20, and through 24 and 24,
to intimate that they are ex-
punged, we have

$10 : 40 :: 12 : 48$

Ex Æquo.

PROPOSITION XIII.

(*See el. 5, Sa. Si. Def. 18, 19.*)

If there be any number of quantities in one series, and as many in another, which being taken (viz. two and two of each series) in order are proportionals, then the extremes of one series shall make a proportion with the extremes of the other. Thus if a, b, c be one series

If $a : b :: d : e$ and $b : c :: e : f$
and d, e, f another series } then shall it be $a : c :: d : f$

For since $a : b :: d : e$ (by sup.) therefore alt. (Pr. 6) $a : d :: b : e$ } whence (Pr. 11) $a : d :: c : f$

Also since (by sup.) $b : c :: e : f$, therefore (alt. (Pr. 6) $b : e :: c : f$ } and alt. (Pr. 6) $a : c :: d : f$

This Proposition often appears in the following form, or in two Proportions, in which one series a, b, c , is found in the two former Ratios, and the other series d, e, f , is found in the two latter Ratios, viz.

$a : b :: d : e$

$b : c :: e : f$

Two such Proportions, therefore, in which the same Quantities are antecedents in one Proportion and Consequents in the other, exhibit a case of this Proposition; and the proportional extremes in the two former and two latter Ratios may be had by striking out those Quantities which are both Antecedents and Consequents. Thus by striking out b, b , and e, e , we have the Proportion already demonstrated, viz. $a : c :: d : f$.—This Case will be found an inference from the demonstration above.

C A S E II.

The demonstration of this Proposition may be extended to as many terms as we please.

Let a, b, c, x , be one series } and let the two contiguous terms of one series be in proportion with
and d, e, f, y , be another series } the two contiguous and corresponding terms of the other series, in
an orderly manner, then also shall it be $a : x :: d : y$

For let the two serieses of four terms be considered as only of three terms each, viz.

a, b, c }
and d, e, f } then by the first case of

this Proposition, $a : c :: d : f$; and since in the two Serieses of four Terms, it is, by supposition, $c : x :: f : y$, we have now two Proportions in the

form above-stated; whence (by this Pr.) $a : x :: d : y$.

D

This

Case 2d.

If 10, 20, 40, 80, one series
and 12, 24, 48, 96, another series

and if

$$10 : 20 :: 12 : 24$$

$$20 : 40 :: 24 : 48$$

$$40 : 80 :: 48 : 96$$

then shall it be

$$10 : 80 :: 12 : 96$$

Ex Aequo.

This second Case often appears in the following form, in which the three former Ratios make one series of four Quantities, and the three latter Ratios make another series of four Quantities.

$$\left. \begin{array}{l} a : b :: d : e \\ b : c :: e : f \\ c : x :: f : y \end{array} \right\} \text{wherefore, by the demonstration}$$

of this Case 2, $a : x :: d : y$

Note. As the number of Quantities has been increased from three to four in each series, so, by the same mode of demonstration, it may be increased from four to five in each series, and so on, *ad infinitum.*

PROPOSITION XIV.

See el 5, Sa. Si. Def. 20.

(*el. 5. Sa. Si. 23.*)

Inordinate Proportion.

If 10, 20, 5 be one series
and 8, 2, 4 be another series

And if $10 : 20 :: 2 : 4$

also $20 : 5 :: 8 : 2$

then expunging 20 & 20, and 2 & 2,

we have $10 : 5 :: 8 : 4$

Ex Aequo perturbate.

If there be two serieses (as in the last Prop.) of equal ratios, which being taken (two and two of each series) *not in the same order*, are proportionals, then also the extremes of one series shall make a proportion with the extremes of the other.

Thus if a, b, c be one series } If $a : b :: e : f$; then shall it be $a : c :: d : f$
and d, e, f another series } and $b : c :: d : e$

For since $a : b :: e : f$ (by sup.) therefore (by Pr. 2) $af = be$ } whence $af = cd$, and (by Pr. 3) $a : c :: d : f$
Also since $b : c :: d : e$ (by sup.) therefore (by Pr. 2) $cd = be$

C A S E II.

The demonstration of this Proposition may be extended to as many terms as we please.

} and let there be an equality of all the ratios of one series to all those in the other, but in a disorderly manner, viz.

Let a, b, c, x be one series
and d, e, f, y be another series }

Let $a : b :: f : y$
and $b : c :: e : f$ } then shall it be
and $c : x :: d : e$ } $a : x :: d : y$

For

For let the two serieses of four terms as above be considered as consisting of only three terms each, viz. a, b, c } and let $a : b :: f : y$ } then by the first case of this Proposition
and e, f, y } and $b : c :: e : f$ }

$a : c :: e : y$ whence (Pr. 2) $ay = ce$ } wherefore $ay = xd$ and consequently
Also by the sup. $c : x :: d : e$ whence (Pr. 2) $xd = ce$ } (by Pr. 3) $a : x :: d : y$

This second Case may also appear in the following form, in which the three former Ratios make one series of four Quantities, and the three latter Ratios make another series of four Quantities.

Case 2d.
If 10, 20, 5, 1 one series
40, 8, 2, 4 another series
and if

$$10 : 20 :: 2 : 4$$

$$20 : 5 :: 8 : 2$$

$$5 : 1 :: 40 : 8$$

then shall it be

$$10 : 1 :: 40 : 4$$

Ex æquo perturbate.

$$\left. \begin{array}{l} a : b :: f : y \\ b : c :: e : f \\ c : x :: d : e \end{array} \right\} \text{wherefore as}$$

before in Prop. 13, $a : x :: d : y$

Note. As each series has been increased from three Quantities to four, so may an increase be made from four to five, and so on, *ad infinitum*.

P R O P O S I T I O N XV.

ft. d. d. th. th. th.
If 1 : 2 :: 3 : 4 and there be a 5 and 6
th. d. th. th.
and if 5 : 2 :: 6 : 4

ft. th. d. d. th. th.
then 1 and 5 : 2 :: 3 and 6 : 4

If $a : b :: c : d$ } then shall $a + e : b :: c + f : d$
and $e : b :: f : d$ }

(el 5, Sa. Si. 24)

If $6 : 2 :: 9 : 3$

and $4 : 2 :: 6 : 3$

Then $4 + 6 : 2 : 9 + 6 : 3$

For since (by sup.) $a : b :: c : d$, therefore (Pr. 2) $ad = bc$ } whence $ad + ed = bc + bf$
Also since (by sup.) $e : b :: f : d$, therefore (Pr. 2) $ed = bf$ } or $a + e \times d = c + f \times b$; and
consequently (Pr. 3) $a + e : b :: c + f : d$

P R O P O .

(*el. 5. Sa. Si. 15.*)

1st—Equimultiples

$$2 : 4 :: 2 \times 10 : 4 \times 10$$

2dly.—Equidivisibles

$$200 : 400 :: \frac{200}{10} : \frac{400}{10}$$

(*el. 5. Sa. Si. Def. 5.*)

Antecedents }
or } multiplied alike
Consequents }

1st—If $2 : 4 :: 3 : 6$

then $2 \times 8 : 4 :: 3 \times 8 : 6$

Or $2 : 4 \times 5 :: 3 : 6 \times 5$

2dly. Antecedents }
or } divided alike
Consequents }

If $16 : 2 :: 24 : 3$

then $\frac{16}{4} : 2 :: \frac{24}{4} : 3$

Or if $2 : 20 :: 3 : 30$

then $2 : \frac{20}{5} :: 3 : \frac{30}{5}$

PROPOSITION XVI.

1st Parts have the same ratio to one another, which their equimultiples have. Thus let a and b be any two homogeneous quantities, whereof ac , bc are equimultiples, then $ac : bc :: a : b$

For $ac \times b = (acb) = bc \times a$; whence (Pr. 3) $ac : bc :: a : b$

2dly. Quantities have the same ratio to one another, which their Equidivisibles have. Thus let a and b be any two homogeneous quantities, and let like parts of these be taken, $\frac{a}{d}$, $\frac{b}{d}$; then $\frac{a}{d} : \frac{b}{d} :: a : b$

For $\frac{a}{d} \times b = \left(\frac{ab}{d}\right) = \frac{b}{d} \times a$; whence (Pr. 3) $\frac{a}{d} : \frac{b}{d} :: a : b$

PROPOSITION XVII.

If four quantities be proportionals, then the first and third, the second and fourth, may be equally multiplied or equally divided. Thus if $a : b :: c : d$, then $ae : b :: ce : d$, also $a : be :: c : de$.

Again $\frac{a}{e} : b :: \frac{c}{e} : d$, also $a : \frac{b}{e} :: c : \frac{d}{e}$

1st. Since $a : b :: c : d$ (by sup.) therefore (Pr. 2) $ad = bc$; wherefore $ad \times e = bc \times e$ Or $ae \times d = b \times ce$ Or $a \times de = be \times c$; from which last two Equations we have (Pr. 3) $ae : b :: ce : d$ And $a : be :: c : de$

2dly. Since (by sup. and Pr. 2) $ad = bc$, therefore $\frac{ad}{e} = \frac{bc}{e}$; Or $\frac{a}{e} \times d = b \times \frac{c}{e}$; Or

$a \times \frac{d}{e} = \frac{b}{e} \times c$; from which last two equations (Pr. 3) $\frac{a}{e} : b :: \frac{c}{e} : d$ And $a : \frac{b}{e} :: c : \frac{d}{e}$

Cor. 1. The first and second; the third and fourth may be equally multiplied or divided. (Alt. by Pr. 6.)

PROPOSITION XVIII.

Any Proportions may be multiplied or divided by one another. Thus if $a : b :: c : d$ and $e : f :: g : h$; then either $ae : bf :: cg : dh$, Or $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$

1st Since

(*el. 5. Sa. Si. 22.*)

*Ex æqualitate rationum ordinate,
or ex æquo ordinate,
or ex æquali.*

If 10, 20, 40 be one series
and 12, 24, 48 be another series
And if $10 : 20 :: 12 : 24$
also $20 : 40 :: 24 : 48$

Then, drawing a line through 20
and 20, and through 24 and 24,
to intimate that they are ex-
punged, we have

$10 : 40 :: 12 : 48$

Ex Æquo.

PROPOSITION XIII.

(*See el. 5, Sa. Si. Def. 18, 19.*)

If there be any number of quantities in one series, and as many in another, which being taken (viz. two and two of each series) in order are proportionals, then the extremes of one series shall make a proportion with the extremes of the other. Thus if a, b, c be one series

and d, e, f another series } If $a : b :: d : e$ and $b : c :: e : f$
then shall it be $a : c :: d : f$

For since $a : b :: d : e$ (by sup.) therefore alt. (Pr. 6) $a : d :: b : e$ } whence (Pr. 11) $a : d :: c : f$
Also since (by sup.) $b : c :: e : f$, therefore (alt. (Pr. 6) $b : e :: c : f$) and alt. (Pr. 6) $a : c :: d : f$

This Proposition often appears in the following form, or in two Proportions, in which one series a, b, c , is found in the two former Ratios, and the other series d, e, f , is found in the two latter Ratios, viz.

$$a : b :: d : e$$

$$b : c :: e : f$$

Two such Proportions, therefore, in which the same Quantities are antecedents in one Proportion and Consequents in the other, exhibit a case of this Proposition; and the proportional extremes in the two former and two latter Ratios may be had by striking out those Quantities which are both Antecedents and Consequents. Thus by striking out b, b , and e, e , we have the Proportion already demonstrated, viz. $a : c :: d : f$.—This Case will be found an inference from the demonstration above.

C A S E II.

The demonstration of this Proposition may be extended to as many terms as we please.

Let a, b, c, x , be one series } and let the two contiguous terms of one series be in proportion with
and d, e, f, y , be another series } the two contiguous and corresponding terms of the other series, in
an orderly manner, then also shall it be $a : x :: d : y$

For let the two serieses of four terms be considered as only of three terms each, viz.

a, b, c } then by the first case of
and d, e, f }

this Proposition, $a : c :: d : f$; and since in the two Serieses of four Terms, it is, by supposition, $c : x :: f : y$, we have now two Proportions in the

form above-stated; whence (by this Pr.) $a : x :: d : y$.

D

This

Case 2d.

If 10, 20, 40, 80, one series
and 12, 24, 48, 96, another series

and if

$$10 : 20 :: 12 : 24$$

$$20 : 40 :: 24 : 48$$

$$40 : 80 :: 48 : 96$$

then shall it be

$$10 : 80 :: 12 : 96$$

Ex Aequo.

This second Case often appears in the following form, in which the three former Ratios make one series of four Quantities, and the three latter Ratios make another series of four Quantities.

$$\left. \begin{array}{l} a : b :: d : e \\ b : c :: e : f \\ c : x :: f : y \end{array} \right\} \text{wherefore, by the demonstration}$$

of this Case 2, $a : x :: d : y$

Note. As the number of Quantities has been increased from three to four in each series, so, by the same mode of demonstration, it may be increased from four to five in each series, and so on, *ad infinitum.*

PROPOSITION XIV.

See el 5, Sa. Si. Def. 20.

(*el. 5. Sa. Si. 23.*)

Inordinate Proportion.

If 10, 20, 5 be one series
and 8, 2, 4 be another series

And if $10 : 20 :: 2 : 4$

also $20 : 5 :: 8 : 2$

then expunging 20 & 20, and 2 & 2,

we have $10 : 5 :: 8 : 4$

Ex Aequo perturbate.

If there be two serieses (as in the last Prop.) of equal ratios, which being taken (two and two of each series) *not in the same order*, are proportionals, then also the extremes of one series shall make a proportion with the extremes of the other.

Thus if a, b, c be one series } If $a : b :: e : f$
and d, e, f another series } and $b : c :: d : e$ then shall it be $a : c :: d : f$

For since $a : b :: e : f$ (by sup.) therefore (by Pr. 2) $af = be$
Also since $b : c :: d : e$ (by sup.) therefore (by Pr. 2) $cd = be$ } whence $af = cd$, and (by Pr. 3) $a : c :: d : f$

C A S E II.

The demonstration of this Proposition may be extended to as many terms as we please.

Let a, b, c, x be one series } and let there be an equality of all the ratios of one series to all
and d, e, f, y be another series } those in the other, but in a disorderly manner, viz.
 $\left. \begin{array}{l} \text{Let } a : b :: f : y \\ \text{and } b : c :: e : f \\ \text{and } c : x :: d : e \end{array} \right\} \text{then shall it be } a : x :: d : y$

For

For let the two serieses of four terms as above be considered as consisting of only three terms each, viz. a, b, c } and let $a : b :: f : y$ } then by the first case of this Proposition
and e, f, y } and $b : c :: e : f$ }

$a : c :: e : y$ whence (Pr. 2) $ay = ce$ } wherefore $ay = xd$ and consequently

Also by the sup. $c : x :: d : e$ whence (Pr. 2) $xd = ce$ } (by Pr. 3) $a : x :: d : y$

This second Case may also appear in the following form, in which the three former Ratios make one series of four Quantities, and the three latter Ratios make another series of four Quantities.

$$\begin{array}{l} a : b :: f : y \\ b : c :: e : f \\ c : x :: d : e \end{array} \left. \vphantom{\begin{array}{l} a : b :: f : y \\ b : c :: e : f \\ c : x :: d : e \end{array}} \right\} \text{wherefore as}$$

before in Prop. 13, $a : x :: d : y$

Note. As each series has been increased from three Quantities to four, so may an increase be made from four to five, and so on, *ad infinitum*.

P R O P O S I T I O N XV.

	ft.	d.	d.	th.		th.	th.
If	1	:	2	::	3	:	4
	th.		d.		th.		th.
and if	5	:	2	::	6	:	4
	ft.	th.	d.	d.	th.	th.	
then	1 and 5	:	2	::	3 and 6	:	4

If $a : b :: c : d$ }
and $e : b :: f : d$ } then shall $a + e : b :: c + f : d$

(*cl* 5, *Sa. Si.* 24)

If $6 : 2 :: 9 : 3$

and $4 : 2 :: 6 : 3$

Then $4 + 6 : 2 : 9 + 6 : 3$

For since (by sup.) $a : b :: c : d$, therefore (Pr. 2) $ad = bc$ } whence $ad + ed = bc + bf$
Also since (by sup.) $e : b :: f : d$, therefore (Pr. 2) $ed = bf$ } or $a + e \times d = c + f \times b$; and
consequently (Pr. 3) $a + e : b :: c + f : d$

P R O P O .

(el. 5. Sa. Si. 15.)

1st—Equimultiples

$$2 : 4 :: 2 \times 10 : 4 \times 10$$

2dly.—Equidivisibles

$$200 : 400 :: \frac{200}{10} : \frac{400}{10}$$

(el. 5. Sa. Si. Def. 5)

Antecedents
or
Consequents } multiplied alike

1st—If $2 : 4 :: 3 : 6$
then $2 \times 8 : 4 :: 3 \times 8 : 6$
Or $2 : 4 \times 5 :: 3 : 6 \times 5$

2dly. Antecedents
or
Consequents } divided alike

If $16 : 2 :: 24 : 3$
then $\frac{16}{4} : 2 :: \frac{24}{4} : 3$
Or if $2 : 20 :: 3 : 30$
then $2 : \frac{20}{5} :: 3 : \frac{30}{5}$

P R O P O S I T I O N XVI.

1st Parts have the same ratio to one another, which their equimultiples have. Thus let a and b be any two homogeneous quantities, whereof ac , bc are equimultiples, then $ac : bc :: a : b$

For $ac \times b = (acb) = bc \times a$; whence (Pr. 3) $ac : bc :: a : b$

2dly. Quantities have the same ratio to one another, which their Equidivisibles have. Thus let a and b be any two homogeneous quantities, and let like parts of these be taken, $\frac{a}{d}$, $\frac{b}{d}$; then $\frac{a}{d} : \frac{b}{d} :: a : b$

For $\frac{a}{d} \times b = \left(\frac{ab}{d}\right) = \frac{b}{d} \times a$; whence (Pr. 3) $\frac{a}{d} : \frac{b}{d} :: a : b$

P R O P O S I T I O N XVII.

If four quantities be proportionals, then the first and third, the second and fourth, may be equally multiplied or equally divided. Thus if $a : b :: c : d$, then $ae : b :: ce : d$, also $a : be :: c : de$.

Again $\frac{a}{e} : b :: \frac{c}{e} : d$, also $a : \frac{b}{e} :: c : \frac{d}{e}$

1st. Since $a : b :: c : d$ (by sup.) therefore (Pr. 2) $ad = bc$; wherefore $ad \times e = bc \times e$ Or $ae \times d = b \times ce$ Or $a \times de = be \times c$; from which last two Equations we have (Pr. 3) $ae : b :: ce : d$ And $a : be :: c : de$

2dly. Since (by sup. and Pr. 2) $ad = bc$, therefore $\frac{ad}{e} = \frac{bc}{e}$; Or $\frac{a}{e} \times d = b \times \frac{c}{e}$; Or $a \times \frac{d}{e} = \frac{b}{e} \times c$; from which last two equations (Pr. 3) $\frac{a}{e} : b :: \frac{c}{e} : d$ And $a : \frac{b}{e} :: c : \frac{d}{e}$

Cor. 1. The first and second; the third and fourth may be equally multiplied or divided. (Alt. by Pr. 6.)

P R O P O S I T I O N XVIII.

Any Proportions may be multiplied or divided by one another. Thus if $a : b :: c : d$ and $e : f :: g : h$; then either $ae : bf :: cg : dh$, Or $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$

1st Since

First Part of Prop. 18.
Proportions multiplied by one another.

If $2 : 4 :: 3 : 6$
and $15 : 5 :: 12 : 4$
then $2 \times 15 : 4 \times 5 :: 3 \times 12 : 6 \times 4$

Second Part of Prop. 18.
Proportions divided by one another.

If $30 : 20 :: 36 : 24$
and $2 : 4 :: 3 : 6$
then $\frac{30}{2} : \frac{20}{4} :: \frac{36}{3} : \frac{24}{6}$

(*el. 6, Si. 22.*)

Cor. 1. If $2 : 4 :: 3 : 6$
 $2 : 4 :: 3 : 6$

then $2 \times 2 : 4 \times 4 :: 3 \times 3 : 6 \times 6$

(*el. 6, Si. 22.*)

Cor. 2. If $4 : 16 :: 9 : 36$
then $\frac{4}{2} : \frac{16}{4} :: \frac{9}{3} : \frac{36}{6}$

1st. Since $a : b :: c : d$ (by sup.); therefore (Pr. 1. cor. 1.) $\frac{a}{b} = \frac{c}{d}$
and since $e : f :: g : h$ (by sup.); therefore (Pr. 1. cor. 1.) $\frac{e}{f} = \frac{g}{h}$

Whence, multiplying both sides by equal fractions (which are by their Nature reducible to the same fraction) we still have an equation, viz.
 $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$ Or
 $\frac{a \times e}{b \times f} = \frac{c \times g}{d \times h}$; therefore
(by Pr. 1) $ae : bf :: cg : dh$.

2dly. $\frac{a}{e} \times \frac{d}{b} = \frac{ad}{eb}$ But the latter sides are equal, for $ad = bc$ (Pr. 2.) because $a : b :: c : d$ (by sup). Also $eh = fg$ (Pr. 2.) because $e : f :: g : h$ (by sup.); consequently
and $\frac{b}{f} \times \frac{c}{g} = \frac{bc}{fg}$ $\frac{a \times d}{e \times b} = \frac{b \times c}{f \times g}$ Or otherwise $\frac{a}{e} \times \frac{d}{b} = \frac{b}{f} \times \frac{c}{g}$; whence (Pr. 3.) $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$

Cor. 1. If four quantities be proportionals, then also are their Squares, Cubes, &c. by the first part of this Prop. For since $a : b :: c : d$ (by sup.) therefore $aa : bb :: cc : dd$ (by this Prop.); and these last being Proportionals, so also are $aa \times a : bb \times b :: cc \times c : dd \times d$, because they are multiplied by a Proportion.

Cor. 2. If Squares, Cubes, &c. be proportionals, then also are their roots, by the second part of this Prop. Or if $aa : bb :: cc : dd$, then (Pr. 2.) $aa \times dd = bb \times cc$. But $\sqrt{aa} \times \sqrt{dd} = \sqrt{bb} \times \sqrt{cc}$, that is $ad = bc$, whence (Pr. 3.) $a : b :: c : d$. So also of Cubes, &c.

Cor. 3. If, in a Proportion, one of the Ratios, or both, consist of fractions, then the numerators of these fractions may be exchanged for terms that are in the same Ratio with them, and so also may the Denominators be exchanged.

First, the Denominators remaining, the numerators may be changed. Thus if $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$ and $c : d :: m : n$, then shall it be $\frac{a}{e} : \frac{b}{f} :: \frac{m}{g} : \frac{n}{h}$

E

For

Cor. 3. First,

If $6 : 12 :: \frac{4}{2} : \frac{16}{4}$ (that is $\frac{16}{8} : \frac{32}{8}$)
 and if $4 : 16 :: 20 : 80$
 then $6 : 12 :: \frac{20}{2} : \frac{80}{4}$
 that is $6 : 12 :: \frac{80}{8} : \frac{160}{8}$

For $m : n :: c : d$ by sup.

And $g : h :: g : h$ for it is the same ratio repeated.

By the 2d part of Pr. 18 $\frac{m}{g} : \frac{n}{h} :: \frac{c}{g} : \frac{d}{h}$ } Wherefore (Pr. 11.) the two former
 Ratios of these two Proportions make
 an Analogy; whence
 But by sup. $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$
 it follows that $\frac{a}{e} : \frac{b}{f} :: \frac{m}{g} : \frac{n}{h}$

Secondly, the Numerators remaining, the Denominators may be changed. Thus if $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$

and if $g : h :: l : o$ then shall it be $\frac{a}{e} : \frac{b}{f} :: \frac{c}{l} : \frac{d}{o}$.

For $c : d :: c : d$ because it is only the same ratio repeated.

And $l : o :: g : h$ by sup.—for the latter ratio of the sup. is here only placed first.

Hence it follows

that $\frac{c}{l} : \frac{d}{o} :: \frac{c}{g} : \frac{d}{h}$ By the 2d part of Prop. 18.

But $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$ by sup.—wherefore (Pr. 11.) the two former Ratios of the two
 last Proportions make an Analogy, viz. $\frac{a}{e} : \frac{b}{f} :: \frac{c}{l} : \frac{d}{o}$.

In both cases, the former Ratio $\frac{a}{e} : \frac{b}{f}$ has remained the same; therefore the Proportion is the same
 after the change as before it. Also if, instead of the fractions $\frac{a}{e} : \frac{b}{f}$ as a former Ratio, we should use
 the whole Quantities $a : b$, the mode of demonstration would be the same.

Cor. 4. To discover, if four given Numbers be Proportionals; let the two first Numbers be
 made Multipliers of the 1st and 3d, and of the 2d and 4th, in such a manner that the two first
 Numbers be multiplied by one another; then their products being necessarily equal, the products
 also

also of the two last Numbers will be equal, if the given Numbers were true Proportionals; because the Multipliers form a Proportion.

Given 2, 7; 6, 21 to find whether these be true Proportionals.

7 2 7 2 These form a Proportion. Wherefore by 1st part

of Prop. 18. $14 : 14 :: 42 : 42$ therefore $2 : 7 :: 6 : 21$, or the four given Numbers were true Proportionals.

Given 2, 5; 7, 17 to find whether these be true Proportionals.

5 2 5 2 These form a Proportion; wherefore by 1st part

of Prop. 18. 10, 10; 35, 34, therefore the given Numbers 2, 5; 7, 17 were not true Proportionals.

Given 3, 2; 11, 8 to find whether these be true Proportionals.

2 3 2 3 These form a Proportion;

By 1st part, Pr. 18. 6, 6; 22, 24 therefore 3, 2; 11, 8 were not true Proportionals.

If we take the four Numbers, which were last given, viz. 3, 2; 11, 8, and multiply the first and the third alike, as by 16, (Pr. 17.) and also multiply the second and fourth alike, as by 23, (Pr. 17.) then we shall have a case of the 7th Definition of the 5th Book of Euclid.

Cor. 4.

(el. 5, Sa. Si. Def. 7.)

Given 3, 2; 11, 8

16 23 16 23 These form a Proportion.

By 1st part, Pr. 18. 48; 46; 176, 184

In the last case, the four given Numbers are multiplied by a Proportion; and therefore, if those four Numbers were true proportionals, then their products would exhibit a true proportion, (by Pr. 18;) but their products are not proportionals, for 48 the multiple of the first term is *greater* than 46 the multiple of the second; but 176 the multiple of the third term is *less* than 184 the multiple of the fourth term;

term; and therefore (by Pr. 18.) the four given Numbers 3, 2; 11, 8 were not proportionals. Further, as it may be observed by the multiples, that 48 hath to 46 a greater ratio, than 176 hath to 184, so also answerably to their multiples, observe of the original ratios, that 3 hath to 2 a greater ratio than 11 hath to 8; for 3 is more than 2 by the half of 2, but 11 is not more than 8 by the half of 8.

OF EUCLID'S TEST OF PROPORTIONALITY.

From Prop. 17, and the cases of this last Cor. 4, we may understand the 5th Definition of Euclid's 5th Book.

Euclid's definition, or test of proportionality, is the Converse of our Prop. 17, and shews, that if the first and third be equally multiplied, and also the second and fourth, and if it be found, that the first and third so multiplied are still alike great in respect of the second and fourth so multiplied, then the four original and unmultiplied Numbers are discovered to be true proportionals; —which is the case of our Cor. 4, of Prop. 18.

In this Definition Euclid makes Equimultiples the test and criterion of proportionality. Such a test is applicable to *incommensurables*, and therefore is the most comprehensive test possible.

And here in the first place we must explain what is meant by *commensurable* and *incommensurable* quantities. Now a less quantity is said to measure a greater, when the less will divide or measure the greater any exact or integral number of times. (Introduct. Sect. 7.). Two quantities then are said to be *commensurable*, when a third quantity can be found that will measure them both; and that third quantity is called their *common measure*. Two quantities are said to be *incommensurable*, when they admit of no common measure.

All *whole numbers* are commensurable; for unity is their common measure. All *fractions* are commensurable, as $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$; for these being reduced to their equivalent fractions, having one common denominator, will be $\frac{70}{105}$, $\frac{84}{105}$, $\frac{90}{105}$. Now it is evident that the fraction $\frac{1}{105}$ will measure every one of them; for it is contained 70 times in the first fraction, 84 times in the second, and 90 times in the third fraction.

All *mixt numbers* are commensurable, for they may be reduced to improper fractions: therefore numbers of all sorts, whether they be whole numbers, *fractional*, or *mixt*, are commensurable.

Two quantities having thus a common measure can be represented by numbers, or are to each other as one number is to another.

Thus the fraction $\frac{70}{105}$ is to the fraction $\frac{84}{105}$ as 70 is to 84, or as 70 Units to 84 Units. The ratio then of Commensurables to each other can be expressed.—On the contrary, the ratio of incommensurables to each other cannot be expressed; for as they admit of no common measure, you cannot say with exactness, how great the one is in respect of the other. We will give an instance of this. The diameter of a circle is incommensurable to the circumference of that circle. If the diameter be 1, the circumference will be $3\frac{1}{4}$ certain decimal parts of that diameter, which parts may be continued *ad infinitum*. Thus if the diameter be 1, the circumference will be 3.1415 &c. *ad infinitum*.

Now as the exact number of decimal parts cannot possibly be expressed, we cannot say exactly, how much the circumference is greater than the diameter, that is, the Ratio of a circumference to it's diameter cannot be expressed in numbers. No number therefore can measure them both; for the number expressing the circumference existeth not; consequently, you cannot measure that which existeth not. For $c : d :: 3.1415 : 1$. Hence we cannot find by our Introduct. Sect. 8, what parts the number expressing the circumference is of the diameter, for a perfect numerator doth not exist. Nor can we, in a proportion in which circumferences and diameters are concerned, apply our Test Prop. 1, Cor. 1. because the fractions arising from the proportion are imperfect, being indeed inexpressible. Nor can we apply our other test of proportionality (Prop. 2), because we cannot multiply that which does not perfectly exist. Thus there is no test in our 5th book, whereby we may prove the truth of that proportion which exists among incommensurables: for our tests apply only to commensurables of the same kind, and such are Units, and all abstract Numbers. But Euclid's test by Equimultiples as stated *el. 5. Sa. Si. Def. 5*, (and throughout his Book V.) will perfectly apply to incommensurables, and will have a regard even to the remaining decimal parts that are not expressed; as for instance, such remaining decimals of a circumference as have been hinted above and are not expressed, being such as might follow after the sign $\frac{1}{4}$. How this may be effected, it is not now our business to inquire; and since our own Tests already laid down are sufficient for our present purpose, it will be right to defer an investigation into the more extensive nature of Euclid's test of proportionality, at least till after the demonstration of *Prop. 4. el. 6*. when *Ludlam's Rudiments of Mathematics* from Sect. 109 to the end of Sect. 116 may be consulted with advantage. We will be content, therefore, now with a right understanding of our own Tests of Proportionality, which are perhaps the easiest, and are drawn not from the 5th, but from the 6th and 7th Books of Euclid; and consequently they are such as Euclid could not use in his 5th Book.—Euclid's Test is suited to that doctrine of Incommensurables which is laid down in his 10th Book; and it has been sufficiently explained for the present in our Prop. 18, Cor. 4; in which we have seen the Euclidean test of four numbers or magnitudes, namely,—that if the 1st and 3d, multiplied alike, and also the 2d and 4th, do produce a proportion in the product or multiples,

F

then

then the four unmultiplied Numbers that were given at first are necessarily proportionals. And this is all that Euclid requires the Young Student to understand before the reading of his future doctrine of Incommensurables, to four of which as proportionals, the said test in Def. 5. may be applied.

Passing by then the extensive use of Euclid's 5th Def. (as we must also pass by at the first at least, the extensive use of many Propositions) we will proceed to explain *Sa. Si. cl. 5. Def. 10 and 11.* This explication requires some preliminary doctrine, which, as it is distinct from the Propositions already laid down, we shall divide into Sections, and consider as a separate work.

OF THE COMPOSITION AND RESOLUTION OF RATIOS.

Sect. 1. From Prop. 18, of our Book V. we learn to compound or put Ratios together; for the Union of two Proportions (that is, the Union of the two former, and of the two latter Ratios in each Proportion) was there made by multiplying Antecedents by Antecedents, and Consequents by Consequents. Upon this principle we shall be able to understand the 4th paragraph of Definition XI. prefixed to Simson's Vth Book of Euclid. This paragraph begins with the words,—And if *A* has to *B* the same ratio which *E* has to *F*, &c.—and it's substance may be represented in the following manner.

If there be a series of quantities, as 48, 40, 30, 15, and if these quantities be made continued Ratios, I say, that the Ratio of the extreme terms will be compounded of the several intermediate Ratios.

48 : 40 } are continued Ratios; and if we divide
40 : 30 } the first Ratio by 8, the second Ratio
30 : 15 } by 10, and the third Ratio by 15, (by
our Pr. 17. Cor. 1.) we shall have three latter (though
not continued) Ratios, equal to the three former Ra-
tios, viz. 48 : 40 :: 6 : 5

40 : 30 :: 4 : 3
30 : 15 :: 2 : 1

Then by Pr. 18.
of our Book V.

If $A : B :: E : F$
and $B : C :: G : H$
and $C : D :: K : L$

Here the three former Ratios are called *continued* Ratios, because in these three Ratios every term, except the first and last, continues to be both consequent of one Ratio and Antecedent of the next Ratio.—The three latter Ratios are not continued Ratios.

$48 \times 40 \times 30 : 40 \times 30 \times 15 :: 6 \times 4 \times 2 : 5 \times 3 \times 1$
or dividing the former Ratio by 40×30 , and performing the operation expressed in the latter Ratio, we have $48 : 15 :: 48 : 15$. Hence the ratio of 48 to 15 contains in itself either the three former or the three latter Ratios; and therefore the Ratio of 48 to 15, is compounded of the ratios of 48 to 40, of 40 to 30, and of 30 to 15; and further, the ratio of 48 to 15, is compounded of the ratios of 6 to 5, of 4 to 3, and 2 to 1.

Then by Prop. 18. of our Book V. $ABC : BCD :: EGK : FHL$; whence dividing the former ratio by BC , Pr. 17. Cor. 1, we have $A : D :: EGK : FHL$

Here the two last Proportions contain one and the same latter Ratio, viz. the Ratio of EGK to FHL , to which the two former Ratios are equal by Def. 4. of our Book V; wherefore those two former Ratios are also equal to one another (by Prop. 11. of our Book V.) and consequently either the Ratio of ABC to BCD , or the Ratio of A

to D equally express the union or composition of the three Ratios of A to B, and B to C, and C to D; and hence the Ratio of A to D is said to be made up of or compounded of these three last named Ratios put together; and these being the three former Ratios in the three Proportions, are severally equal to the three latter Ratios (by Def. 4. of our Book V;) so that the Ratio of A to D is also compounded in effect of the three Ratios of E to F, G to H, and K to L.

Again—If $96 : 30 :: 48 : 15$, then the ratio of 96 to 30 is, in like manner compounded of the three former, or the three latter Ratios produced in the last marginal Note

Again, if $M : N :: A : D$, then also is the ratio of M to N compounded of the three former or of the three latter ratios. [See the last Paragraph of Simson's Def. 11. el. 5.]

This Composition of Ratios is by some called an Addition to Ratios.

From what has been laid down above, we shall easily understand the definition of a Compound Ratio.

DEFINITION OF A COMPOUND RATIO.

Sect. 2. A Ratio is said to be compounded of two (or more) Ratios, when the terms of those Ratios multiplied together do produce such Ratio. Thus since any Ratio may be expressed fraction-wise, (Introduct. Sect. 8). the fraction $\frac{ab}{cd}$ expresses the ratio of ab to cd , which is a compound ratio, containing the two ratios of a to c , and b to d united; for $\frac{ab}{cd} = \frac{a}{c} \times \frac{b}{d}$. Thus the ratio of 24 to 1 may be said to be compounded of the two ratios of 2 to 1 and of 12 to 1; because $\frac{24}{1} = \frac{2}{1} \times \frac{12}{1}$. Thus also, if there be two right angled Parallelograms, and if one of these have 2 Inches in breadth, and the other 1, and if the former have 12 inches in length, and the latter 1; then on account of the breadth, the area of the greater figure will be to the area of the less figure as 2 to 1; and on account of the Length, the greater area will be to the less as 12 to 1; wherefore on account both of breadth and length, the greater area will be to the less in a ratio compounded of 2 to 1, and of 12 to 1; that is of 24 to 1.

DEFINITION OF CONTINUED RATIOS.

Sect. 3. Continued Ratios are those in which every term (except the first and last) continues to be the Consequent of one Ratio and the Antecedent of the next Ratio. Thus the series of quantities 48, 40, 30, 15, may be made to exhibit continued Ratios, viz.

$$48 : 40$$

$$40 : 30$$

$$30 : 15. \text{--- See Sect. 1. and the figures there in the margin.}$$

IN

IN CONTINUED RATIOS, THE RATIO OF EXTREMES IS EQUAL TO ALL THE INTERMEDIATE RATIOS.

Sect. 4. In a series of quantities, a, b, c, d, e , if they be made to exhibit continued ratios, the Ratio of the extremes is compounded of all the intermediate Ratios; that is, the ratio of a to e is compounded of the ratio of a to b , b to c , c to d , and d to e , which are all the intermediate and continued ratios between a and e . For all these intermediate ones may (by our Introduct. *Sect. 8.*) be thus expressed, $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}$; and the ratio compounded of these intermediates is $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{abcd}{bcde} = \frac{a}{e}$. Therefore $\frac{a}{e}$ or the ratio of a to e = all the intermediate Ratios.

To exemplify this by Numbers, let a and e be represented by 48 and 2, and between 48 and 2 let any Number of any terms whatsoever be arbitrarily inserted, as 24, 6, 18, 36, 6, 60; then, if all these terms be made to exhibit *continued* ratios (that is, if all the terms except 48 and 2, the first and last, be made both consequents and antecedents) I say, that the ratio of 48 to 2 will contain in itself all the intermediate ratios put together; that is,

$$\begin{array}{l} \text{The ratio of 48 to 2 is compounded of the Ratios of 48 to} \\ \quad 24, \text{ of 24 to 6, of 6 to 18, of 18 to 36, of 36 to 6, of} \\ \quad 6 \text{ to 60, and of 60 to 2.} \end{array} \left\{ \begin{array}{l} 48 : 24 \\ 24 : 6 \\ 6 : 18 \\ 18 : 36 \\ 36 : 6 \\ 6 : 60 \\ 60 : 2 \end{array} \right.$$

For since all these ratios may be represented by fractions, it follows that the ratio of 48 to 24 will become $\frac{48}{24}$, and so also every other ratio may be turned into a fraction; whence all the ratios put together will be,

$$\frac{48}{24} \times \frac{24}{6} \times \frac{6}{18} \times \frac{18}{36} \times \frac{36}{6} \times \frac{6}{60} \times \frac{60}{2} = \frac{48 \times 24 \times 6 \times 18 \times 36 \times 6 \times 60}{24 \times 6 \times 18 \times 36 \times 6 \times 60 \times 2} = \frac{48}{2}$$

and hence the parts or component Ratios may be said to equal one whole.

That part of *Simson's Def. 11, prefixed to Euclid, Book V.* which begins with the words, "*Definition A, to wit, &c.*" may now be read; and it's contents will be found stated in the three former Ratios of those Three Proportions (and inferences) that have been produced in our *Sect. 1.*

To

To understand the case above fundamentally, we must consider how the various multiplications of 48 tend to increase 48; and how, on the other hand, the various and like multiplications of 2 diminish such increase, by multiplying 2; whereby we shall find, that the given ratio is finally left as it was at first found, viz. the ratio of 48 to 2.

HOW ANY RATIO IS MADE GREATER BY INCREASING IT'S ANTECEDENT.

Sect. 5. Any ratio is said to be greater than another, according as its Antecedent hath a greater proportion to it's Consequent than the other Ratio's Antecedent hath to it's Consequent. Thus in comparing the ratio of 12 to 1, with the ratio of 2 to 1, I observe, that 12 contains it's Consequent 12 times, whereas 2 contains it's consequent only twice, wherefore the ratio of 12 to 1 is a greater Ratio than that of 2 to 1. And since any Ratios may be represented by Fractions (*Introduct. Sect. 8.*) we can easily see in what proportion the one Ratio is greater than the other. For,

$$\begin{aligned} \text{The Ratio of 12 to 1 : Ratio of 2 to 1} &:: \frac{12}{1} : \frac{2}{1} \text{ or} \\ &:: 12 : 2 \text{ or (by Pr. 17. Cor. 1. of our Book V.) dividing by 2,} \\ &:: 6 : 1 \end{aligned}$$

Hence we perceive, that the ratio of 12 to 1 hath 6 times that magnitude which 2 to 1 hath; for 12 is 6 Doubles of its Consequent, and 2 is only one Double of it's Consequent 1. Thus the measure of a Ratio's greatness is made by estimating the greatness of it's antecedent in respect of it's Consequent; and thus a Ratio is made *greater* by increasing it's *Antecedent*. Hence, therefore, we determine, how *great* one Ratio is in respect of another, but not *how oft* one Ratio is contained in another, as will be seen hereafter.

HOW ANY RATIO IS MADE LESS BY INCREASING IT'S CONSEQUENT.

Sect. 6. From what has been said, it follows that the less the Antecedent of a Ratio is in respect of it's Consequent, the less is the Ratio itself; or, in other words, the more the Consequent is increased, the less must be the comparative magnitude of the Antecedent. Thus, if the ratio of 1 to 1 become the ratio of 1 to 3, this latter ratio hath but a third of the magnitude of the former Ratio, (by *Sect. 5.*) and thus a Ratio is made *less* by increasing it's *Consequent*.

Sect. 7. Hence, in RESOLVING A RATIO, we may use arbitrarily greater or less Ratios than that resolved, if we use *continued* Ratios in the Resolution, because *continued* Ratios multiply alike the Antecedent and Consequent of the Resolved Ratio; whence it follows, that though they increase the magnitude of the Resolved Ratio by multiplying it's Antecedent, they equally diminish the magnitude of the Resolved Ratio by multiplying it's Consequent, and thereby leave it unaffected.

Thus the ratio of 2 to 1 may be resolved into the two ratios of 2 to 6, and of 6 to 1 ; the former of which is just as much less than the resolved Ratio as the latter is greater than it. For first compare the latter Ratio with it, viz.

The ratio of 6 to 1 : resolved ratio of 2 to 1 :: $\frac{6}{1}$: $\frac{2}{1}$ or

$\therefore 6 : 2$ or (by Prop. 17. Cor. 1. of our Book V.) dividing by 2,

$$:: 3 : 1$$

Here it appears, that one of the parts, (viz. 6 to 1) into which we have resolved the ratio of 2 to 1, hath 3 times as great an Antecedent as the Resolved Ratio; wherefore, if compounded with the Resolved Ratio, it would increase it's Antecedent 3 times.—On the other hand, the other part of the resolution, or the ratio of 2 to 6, having a Consequent 3 times it's Antecedent, will diminish the Ratio it is compounded with 3 times; viz.

* Divide this former Ratio by 2 to produce the latter Ratio.

$$*_2 : 6 :: 1 : 3$$

Here the Consequent will diminish the resolved Ratio three times, if compounded with it.

$$6 : 1 :: 6 : 1$$

Here the Antecedent will increase the resolved Ratio three times, if compounded with it.

Compounding by the same }
method as in *Sec.* 1. }

$\} 2 : 1 :: 6 : 3$ or (by Pr. 17. Cor. 1 of our Book V.) dividing by 3,

$$:: 2 : 1$$

The continued ratios of 2 to 6 and of 6 to 1 are the parts into which the ratio of 2 to 1 is resolved; and these parts taken together do equal their whole (Sect. 4.); wherefore the ratio of 2 to 1 may be so resolved; and it will often be found more convenient to use the parts than the whole undivided Ratio.

Sec. 8. In like manner any number of continued Ratios may be inserted between 2 and 1. For

+ Divide the former Ratio by 2.

$$+2 : 6 :: 1 : 3$$

‡ Divide the former Ratio by 6 to produce the latter Ratio.

$$\begin{array}{l} \frac{1}{2} 6 : 12 :: 1 : 2 \\ 12 : 1 :: 12 : 1 \end{array}$$

In this case the ratio of 2 to 1 is resolved into the ratios of 2 to 6, of 6 to 12, and of 12 to 1; and all these parts or ratios being compounded as in *Seet. 7.* will exhibit the ratio of 2 to 1; that is, all these parts taken together do equal or constitute their whole (*Seet. 4.*); wherefore the ratio of 2 to 1 may be so resolved.

WHY THE RATIO MAY BE SO RESOLVED.

The last of the three Ratios used in Resolution (viz. that of 12 to 1.) has six times the greatness which 2 has to 1; for 2 is one double of 1, and 12 is 6 doubles of 1; and consequently this last Ratio of 12 to 1 would, in composition, increase the Ratio of 2 to 1 six times; but the other two Ratios having 2 times 3 (or 6) for their Consequents (*See the latter Ratios in the Proportions above,*) will diminish the ratio of 2 to 1, in composition six times. Hence we see the reason, why the three component parts or Ratios being taken together must constitute the whole ratio of 2 to 1, since the parts of resolution do as much increase the whole ratio of 2 to 1 in one respect as they diminish it in another.

Sect. 9. The same will be the case, whatever be the Ratios, into which we arbitrarily resolve any Ratio, provided such Ratios be continued Ratios.

Resolve the Ratio of 48 to 2 into any number of continued Ratios and let some of these Ratios be greater, and others less than the Resolved Ratio.

24)	48 : 24 ::	2 :	1	} In this case the ratio of 48 to 2 is resolved into the ratios of 48 to 24, of 24 to 6, of 6 to 18, of 18 to 36, of 36 to 6, of 6 to 60, and of 60 to 2.—All the latter Ratios are produced by dividing each former Ratio, by such Figure as is set before it.
6)	24 : 6 ::	4 :	1	
6)	6 : 18 ::	1 :	3	
9)	18 : 36 ::	2 :	4	
6)	36 : 6 ::	6 :	1	
6)	6 : 60 ::	1 :	10	
2)	60 : 2 ::	30 :	1	

Compounding by the same method as in Sect. 1. } $48 : 2 :: 2880 : 120$, or, dividing by 10,
 $:: 288 : 12$, or, dividing by 12,
 $:: 24 : 1$

Here the ratio of 48 to 2 is compounded of all the former, or of all the latter Ratios. It is compounded of the former ratios because as much as their Antecedents would increase the Ratio resolved by multiplying 48, so much do the same multipliers (being also Consequents) diminish the Resolved Ratio by multiplying its Consequent 2.—It is also compounded of the latter Ratios, because they are equal to the former Ratios.—Observe also of these latter Ratios, that they so increase and diminish one another as in the end to produce a Ratio equal to that of 48 to 2.

The

The ratios that *increase*, are

2 to 1
4 to 1
6 to 1
30 to 1

} These Ratios, by having Great comparatives Antecedents, tend to an increase in Compounding. Sect. 5.

The total increasing power of these is by } 1440 : 1
compounding found to be the ratio of

The ratios that *diminish* are

1 to 3
2 to 4
1 to 10

} These Ratios, by having great comparative Consequents, tend to a Decrease in compounding. Sect. 6.

The total diminishing power of these is by } 2 : 120, or, dividing by 2,
compounding found to be the ratio of } 1 : 60

Hence,

Increase : Decrease :: 1440 : 60, or, dividing by 60,

:: 24 : 1

Thus all the latter Ratios upon the whole increase one another's Antecedents, and they so increase each other, that 24 is the final Magnitude of all the Antecedents; and the same latter ratios, by increasing one another's Consequents, do so diminish one another, that 1 is the final magnitude of all the consequents; whereby the ratio of 48 to 2 (= Ratio of 24 : 1) has been resolved into such parts as make it up when put together; and therefore it may rightly be so resolved, because such parts being taken together do equal their whole, that is, all the component ratios put together finally equal the ratio of 48 to 2; and thus we may fully and fundamentally understand WHY the ratio of 48 to 2 may be so resolved; as we promised to shew at the end of Sect. 4.—Now in considerations of Natural Philosophy, the component Ratios will often be more useful than the whole unresolved and original Ratio.*

Sect. 10.

* Since it has now been fully seen, that increasing or multiplying the Antecedent of a Ratio makes the Ratio greater, and that the like multiplication of it's Consequent makes the Ratio just so much less, whereby upon the whole the Ratio is left unchanged, we have hence an easy method of comparing Ratios whose Antecedents and Consequents are both different, viz. by multiplying the terms of each ratio by the Consequent of the other. Let it required to compare the ratio of 5 to 4 with the ratio of 7 to 3, and to determine which may be the greater Ratio. Then

By Prop. 16 of } 5 : 4 :: 5 × 3 : 4 × 3, or, as 15 to 12,
our Book V. } 7 : 3 :: 7 × 4 : 3 × 4, or, as 28 to 12.

The latter Ratios in the Proportions above are equal to the former Ratios; but the latter Ratio must have the same Consequent, because such Consequent is

Sec. 10. We may now understand *Simson's Euclid*, or *Saunderson*, *Book V. Def. 10.*—But first, as Euclid in that Definition is speaking of continued proportionals, we will define them.

Four Quantities are necessary to form a Proportion, which four are called Proportionals; but by *Def. 5.* of our *Book V.* Three Quantities may be Proportionals; but these must be continued Proportionals, in which the ratio between any term and it's next subsequent one continues the same throughout. Thus, 2, 4, 8, are continued Proportionals, in any two of which, taken in the order in which they stand, or as succeeding terms, (as *Def. 5.* of our *Book v.* expresses it) there is the same ratio of 1 to 2.

SIMSON'S EUCLID, OR SAUNDERSON, BOOK, V. DEF. 10.

When three magnitudes are Proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

For let a, b, c , be three continued Proportionals; then, because the ratio between each term and it's next subsequent one continues the same throughout, we shall hence have the following proportions.

If 2, 4, 8 be continued Proportionals,

Then $2 : 4 :: 2 : 4$

Also $4 : 8 :: 2 : 4$

$a : b :: a : b$. Here the first and second are compared.

Also $b : c :: a : b$. Here the second and third are compared.

$2 \times 4 : 8 \times 4 :: 2 \times 2 : 4 \times 4$
Dividing the former Ratio by 4,
 $2 : 8 :: 4 : 16$

By 1st part, *Pr. 18.* our *Book v.* $ab : cb :: a^2 : b^2$, and dividing the former ratio by b , we have,

By *Pr. 17. Cor. 1.* our *Book v.* $a : c :: a^2 : b^2$ Here

the product of the two Consequents of the given ratios. It may now be seen then, that the Ratio of 7 to 3 is the greater ratio; because 28 hath a greater magnitude when compared with 12, than 15 hath when compared with the same 12. [*El. 5. Sa. Si. Prop. 8*]

Hence it appears that the ratio of 28 to 12, that is the ratio of 7 to 3 is the greater ratio according to our *Sec. 5.* and *Euclid's el. 5. Sa. Si. Prop. 8.* which *Prop.* may now be understood.—That the ratio of 7 to 3 is the greater ratio might indeed be perceived at once by our *Sec. 5*; for, since the greater the Antecedent is in respect of it's Consequent, the greater is the ratio, it may be observed, that the Antecedent 7 contains its Consequent 3 twice and $\frac{1}{3}$ of it also, whereas the Antecedent 5 contains it's Consequent 4 only once and $\frac{1}{4}$ of it besides. The same thing may be observed of their representative ratios, viz. the ratio of 28 to 12, and of 15 to 12. Thus then the ratio of 28 to 12, hath by it's Antecedent nearly a double magnitude in respect of the ratio of 15 to 12, but it is not therefore nearly twice the ratio of 15 to 12, or two repetitions of the ratio of 15 to 12; for such doubled ratio of 15 to 12, or (which is the same thing) of 5 to 4 can be only had by compounding the Antecedent repeated, and the Consequent repeated, viz. $5 \times 5 : 4 \times 4$, or the ratio of 25 to 16; as will be seen hereafter. Thus a stone may, in point of magnitude, be the double of the magnitude of a bit of lead, and yet the stone is not that identical lead twice taken.—Now as to the exact part that the lesser Ratio of 5 to 4 is of the greater Ratio of 7 to 3 we cannot express it, though it be certainly some part, because these two ratios (like most other ratios, but not all) are incommensurable, that is, no common ratio can be found that will measure them both, as may be seen hereafter in *Saunderson's Sec. 294*—This observation has been made now merely to prevent future misconception; and with the same view we shall further observe, that though the fraction $\frac{28}{12}$ (which represents the ratio of 7 to 3) is nearly double of the fraction $\frac{15}{12}$ (which represents the ratio of 5 to 4) yet the ratio of 7 to 3 is not nearly double of the ratio of 5 to 4; for all the properties of fractions are not applicable to them when considered as the Representatives of ratios, because fractions being representatives of Ratios increase not in the same proportion with the increasing Ratios represented by them. Thus the double of the fraction $\frac{15}{12}$ is $\frac{30}{12}$, but the double of the ratio of 15 to 12 would be the ratio of 15×15 to 12×12 , or the ratio of 225 to 144, or (dividing both terms by 9) of 25 to 16, which is very different from that ratio of 30 to 12 which would arise from the doubled fraction just stated.—Thus we have seen, that neither doth a ratio, because it's Antecedent hath a double magnitude in respect of a less Ratio, imply, that it is the double of both terms of that less ratio; nor does a fraction, because it is the double of a less fraction, imply, that the ratio arising from the doubled fraction is the double of the ratio arising from the less fraction.

Here the first, if compared with the third, is as great as the two ratios taken together which intervene between the first and the third; but these two ratios are the same with that of a to b ; and therefore the ratio of aa to bb is called the duplicate of the ratio of a to b or the ratio of a to b twice taken.—The very Process of Composition, as stated in this operation, shews that twice the ratio of a to b is the ratio of aa to bb ; and therefore half the ratio of aa to bb is it's square root, or the Ratio of a to b . Thus half any ratio is the square root of that Ratio. Half the ratio of 25 to 36 is the ratio of 5 to 6; and half the ratio of a to b is the ratio of $\sqrt{a}$: $\sqrt{b}$.

Note.—We may now understand the latter part of *Saunderson's Def. 10. Book v.* For since, as has been demonstrated, $a : c :: a^2 : b^2$, let the latter ratio of this proportion first, viz.

$$a^2 : b^2 :: a : c, \text{ wherefore, by our Book v.}$$

Pr. 18. Cor. 2. $a : b :: \sqrt{a} : \sqrt{c}$, or the first is to the second as the square root of the first to the square root of the third, or (as it is otherwise expressed) in a subduplicate ratio of the first to the third.—Here observe, that as the duplicate of a ratio is the same ratio twice taken and united, or the square of that ratio, so the subduplicate of a ratio is it's square root, or the half of that ratio, or one part out of two. It is derived from sub under; it is one of the under or inferior parts of the duplicate, since all that which is contained under the duplicate is the two halves (as it were) composing the whole of the duplicate ratio. The Sub-duplicate ratio therefore is one of the Contents of the Duplicate Ratio.

SIMSON'S EUCLID, OR SAUNDERSON, BOOK V. DEF. 11.

Sett. 11. When four magnitudes are continual Proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on to quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

Let a, b, c, d , be continued proportionals, and by the same way of reasoning as in *Sett. 10*, compare the first with the fourth term. Then, because the ratio between any term and it's next subsequent one continues the same throughout, we shall hence have the three following proportions.

If 2, 4, 8, 16, be continued Proportionals, then			
2 :	4 ::	2 :	4
and 4 :	8 ::	2 :	4
and 8 :	16 ::	2 :	4
$2 \times 4 \times 8 :$	$16 \times 4 \times 8 ::$	$2^c :$	4^c
or 2 :	16 ::	8 :	64

$$\begin{array}{lcl} a : b :: a : b & \left\{ \begin{array}{l} \text{the 1st and 2d of the Four compared} \\ \text{with themselves.} \end{array} \right. \\ b : c :: a : b & \left\{ \begin{array}{l} \text{the 2d and 3d of the Four compared} \\ \text{with the 1st and 2d, as before.} \end{array} \right. \\ c : d :: a : b & \left\{ \begin{array}{l} \text{the 3d and 4th compared as be-} \\ \text{fore.} \end{array} \right. \end{array}$$

By *Pr. 18.* of our Book v. $abc : dbc :: a^3 : b^3$, { and by dividing the former ratio by bc , we have

$$a : d :: a^3 : b^3 \quad \text{In}$$

In the last proportion it is shewn, that the first is to the fourth as the cube of the first to the cube of the second; or, since these cubes are made up of three ratios of a to b , the first is as great when compared with the fourth, as three ratios of a to b taken together, or as a triplicate ratio of a to b .

Note. In the case of four continued proportionals, the word triplicate at the same time shews that there are three equal ratios intervening between the first and fourth; and that the comparison of the first with the fourth equals the comparison of all those three ratios put together. The same kind of observation may be made of the word *Duplicate*, viz. that it shews two ratios intervene between the first and third term; and that the first, if compared with the third, has the same magnitude as those two ratios put together.

From the process of the demonstrations in this and the preceding Section it will appear, that a ratio is doubled by squaring it's terms, and tripled by cubing it's terms, and thus a ratio may be said to be multiplied by 2 or by 3. Such a composition of Ratios is by some called a multiplication of Ratios. In like manner to extract the square root of a duplicate Ratio is to take the half of that ratio, or, as it were to divide it by 2; and to extract the Cube root of a triplicate ratio is to take a third of that ratio, or, as it were to divide it by 3. Thus the extraction of the roots of Ratios answers to the idea of a Division of Ratio, and by some is so called. For instance, a third of the ratio of 125 to 216 is the cube root of it's terms, or the ratio of 5 : 6.

QUALITY BELONGS NOT TO PROPORTION.

Señ. 12. Before we conclude our 5th Book, let us observe, (from Mr. Ludlam's Rudiments of Mathematics, page 21,) that all questions about proportion have reference to magnitude only, and not to quality. The idea of proportion results from contemplating and comparing the magnitudes only of quantities; their quality has nothing to do with it: and, therefore, if the signs $+$ and $-$, and the distinction of quantities into affirmative and negative is arbitrarily retained, such distinction is not warranted by the subject itself. The following case therefore is absurd,

$$- 1 : + 1 :: + 1 : - 1$$

This proportion, say some, is demonstrable, because the two extreme terms multiplied will equal the product of the two mean terms. But all this is nonsense. In respect of magnitude, $+ 1$ and $- 1$ are equal; and it is in respect to their magnitude only that they can have proportion to each other. Proportion has no concern with their quality. They have no quality in a question about their proportion only.

SIMSON'S EUCLID. BOOK VI. PROPOSITION I.—SEE SIMSON'S FIGURE.

To facilitate the further progress of the Student, it may not be amiss to add here Mr. Ludlam's demonstration of the first Proposition of Euclid's Sixth Book. This demonstration is restrained indeed to commensurables, but it is perhaps on that very account easier to be understood by learners.

“Let the triangles ACH and ADL have the same altitude, viz. the perpendicular drawn from the point A to BD; (*supposing such a perpendicular to be drawn in Simson's Figure*) then as the base CH is to the base DL, so is the triangle ACH to the triangle ADL.

Let the base CH be commensurable to the base DL, and let CB be their common measure. Divide the base CH by the common measure into the equal parts CB, BG, GH; also divide the base DL by the same common measure into the parts DK, KI, equal to one another, and to each of the parts of the base CH. [*Suppose Euclid's Figure to be so constructed, or draw one according to Mr. Ludlam's directions.*] From the vertex A, draw lines to each of the points of division in the base CH; that is, draw AB and AG, and the great triangle ACH will be divided into as many less triangles, as there are parts in its base CH. In like manner, draw lines to each of the points of division in the base DL (viz. draw AK) and this triangle also will be divided into as many less triangles, as there are parts in its base DL. Now all these less triangles, in each of the two greater ones, are equal to one another by 38. 1. El. Therefore the whole triangle ACH will be to the whole triangle ADL, as the number of little triangles which make up the former, to the number of little triangles which make up the latter; that is, as the number of equal parts (or lines) which make up the base of the former triangle, to the number of (the like) equal parts which make up the base of the latter; that is, as the whole base CH to the whole base DL.”

It has now been proved, that, as the base CH is to the base DL, so is the triangle ACH to the triangle ADL; so that if the base CH be *equal* to the base DL, then would the triangle ACH be *equal* to the triangle ADL; if the base CH be *greater* than the base DL, then would the triangle ACH be *greater* than the triangle ADL; if the base CH be *less* than the base DL, then would the triangle ACH be *less* than the triangle ADL. Wherefore these four Quantities, viz. the two Bases and the two triangles, have the very mark and character of proportionality laid down in demonstrating Prop. 1, of our Book V; and consequently they are proportionals.

We may now apply the foregoing demonstration to the triangles ABC and ACD in Simson's Figure, for these triangles have the same perpendicular altitude. Wherefore, by the foregoing demonstration, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

The latter part of the demonstration of this 1st Prop. beginning with “And because the parallelogram CE is double, &c.” may be easily understood in Simson's Euclid; and its substance may be thus represented.

$$BC : CD [:: ABC : ACD] :: EC : CF$$

$$\text{or } 2 \times ABC : 2 \times ACD$$

F I N I S.



